Two locally convergent path following controllers:

Two Path Following Controllers Based On Geometric Argument

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Chapter 1

Problem

1. x denotes x coordinate of AGV position and y denotes y coordinate of AGV position in a global reference frame.
2. p denotes (x, y, θ), v denotes speed, that is $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ and $a = \dot{v}$.
3. θ denotes heading vector, $z(\theta)$ denotes the corresponding unit length complex number. α denotes vector $(\cos\theta, \sin\theta)$. θ_e denotes heading error with clockwise heading error being negative, and ω denotes heading rate, that is, $\omega = \dot{\theta}$.
4. δ denotes steering angle, and $u = (\delta, q)$ denotes control input, where q is input for system $\dot{v} = g(v, \delta, q)$, which is called the dynamic model of the AGV.
5. t denotes physical time, and $\bar{t}$ denotes a dimensionless generic variable. Without loss of generality, we assume that both of them only take nonnegative values.
6. s denotes length variable of a curve, and $\kappa(s)$ denotes **signed** curvature of a curve at length s. We follow the convention that s increases when an AGV along the path has a positive speed (moving forwardly), and along the direction of increasing s, counter-clockwise turning means $\kappa(s) > 0$ and clockwise turning means $\kappa(s) < 0$.
7. let L denotes the length of the main axis of the AGV, that is, L is the distance between the center of rear wheel axis and the center of frontal wheel axis.

Problem 1 (**Path tracking**). *Given the controlled dynamic of the AGV as* $\dot{p} = f(p, u)$ *with respect to physical time* t*, a reference path* $\tilde{p}(\bar{t})$ *with respect to dimensionless variable* $\bar{t}$*, and a reference speed* $\tilde{v}(\bar{t})$*, find a feedback control law* $u = u(p)$ *and a differentiable* $\mathrm{n}(t) = \bar{t}$ *such that solution to* $\dot{p} = \mathrm{f}(p, u)$ *satisfies the following:*

1. *for any* $\varepsilon > 0$ *and* $t_1 \geq 0$*, there exists a* $\delta(t_1) > 0$ *such that* $\|p(t_1) - \tilde{p}(n(t_1))\| < \delta$ *implies* $\|p(t_2) - \tilde{p}(n(t_2))\| < \varepsilon$.
2. $\lim_{t\to\infty} \|p(t) - \tilde{p}(n(t))\| = 0$.

3. $\lim_{t\to\infty} |\tilde{v}(n(t)) - v(t)| = 0$.

If the last requirement (requirement 3) of the above definition is excluded, the problem turns into a **trajectory tracking** problem. If δ in requirement 1 is independent of t_1, the system is called being uniformly (asymptotically) stable. If the decay rate is bounded above by a exponentially diminishing function, it is called being exponentially stable.

When the reference speed $\tilde{v}(\tilde{t})$ is varying slowly, and the speed of the AGV is controlled by an independent controller. We can assume that the error between nominal speed and actual speed is always below a small threshold. In this case, **Path Tracking** can be achieved with a **trajectory tracking** controller with a good enough speed controller. This is the approach we adopt in this project.

Chapter 2

The Kinematic Bicycle Model

We start from a bicycle model with its main axis length being L. The actual control needed to control the AGV is left out as an implementation detail by defining a mapping from control input to the bicycle model to control input to the actual model. In bicycle model, the contacting point of the rear wheel with the ground is denoted by $p_r := (x_r, y_r)$, and that of the frontal wheel is denoted by $p_f := (x_f, y_f)$. We have

$$(\dot{p}_r \cdot \hat{e}_y)\cos(\theta) - (\dot{p}_r \cdot \hat{e}_x)\sin(\theta) = 0 \tag{2.1}$$

This is called no-slippery condition of the rear wheel. Similarly, there is no-slippery condition for the frontal wheel:

$$(\dot{p}_f \cdot \hat{e}_y)\cos(\theta + \delta) - (\dot{p}_f \cdot \hat{e}_x)\sin(\theta + \delta) = 0 \tag{2.2}$$

where $\hat{e}_x$ and $\hat{e}_y$ are unit vector along x direction and unit vector along y direction in the world coordinates. If we define $v_r := \frac{\dot{p}_r \cdot (p_f - p_r)}{\|p_f - p_r\|}$, we can get the following equivalent form, notice that $\frac{\tan\delta}{L} = \frac{1}{R_r}$, where R_r is the turning radius of the rear wheel.

$$\begin{cases} \dot{x}_r &= v_r \cos\theta \\ \dot{y}_r &= v_r \sin\theta \\ \dot{\theta} &= \frac{v_r}{L}\tan\delta \end{cases} \tag{2.3}$$

Similarly, we can get the kinetic constraints in terms of frontal wheel.

$$\begin{cases} \dot{x}_f &= v_f \cos(\theta + \delta) \\ \dot{y}_f &= v_f \sin(\theta + \delta) \\ \dot{\theta} &= \frac{v_f}{L}\sin\delta \end{cases} \tag{2.4}$$

Notice that with the frontal wheel $\frac{\sin\delta}{L} = \frac{1}{R_f}$, where R_f is the turning radius of the frontal wheel. With both wheels, a positive radius (a positive δ) means turning counter-clockwise with positive speed, or turning clockwise with a negative speed. Two consequences from the above two sets of equations are:

$$\frac{v_r}{v_f} = \cos\delta \tag{2.5}$$

$$\delta = \arctan\frac{L\omega}{v_r} \tag{2.6}$$

Given a pre-specified path, for any point $p = (x, y, \theta)$, we use $\tilde{p} = (\tilde{x}, \tilde{y}, \tilde{\theta})$ to denote the nearest point on the path with respect to p. Note that this relationship is only well-defined if there is only one nearest point with a given point. This assumption always holds if we assume that the AGV can only slightly deviate from the pre-specified path, and the path is smooth enough.

2.1 Rear Wheel Controller

First, we define the following symbols, where $\times$ between two vectors are their cross product, that is, $d \times \alpha = \det(\{d, \alpha\})$.

$$\tilde{\alpha} := (\cos\tilde{\theta}, \sin\tilde{\theta}) \tag{2.7}$$

$$d := p - \tilde{p} \tag{2.8}$$

$$e := d \times \tilde{\alpha} = d_x \sin\tilde{\theta} - d_y \cos\tilde{\theta} \tag{2.9}$$

Notice that if the AGV is on the right hand side of the path, e will be positive, and if the AGV is on the left hand side of the path, e will be negative, we define the heading error θ_e of the AGV as

$$z(\theta_e) = \frac{z(\theta)}{z(\tilde{\theta})} \tag{2.10}$$

Recall that $n(t)$ is defined as a mapping from physical time t to a dimensionless curve parameter $\bar{t}$. It is clear that the mapping from any p to $\tilde{p}$ implicitly defines such an $n(t)$ too.

From the perspective of t, it is obvious that

$$v_r \cos\theta_e = w|R_p| \tag{2.11}$$

where R_p is the **signed** curvature radius at point p along the actual path, and ω is the heading rate of the AGV at the same point. (To simplify expression, we assume there is a virtual AGV with its state being given as $\tilde{p}$.) From the perspective of $\bar{t} = n(t)$, the "speed" of the virtual AGV with respect to nominal time $\bar{t}$ is $\frac{ds}{d\bar{t}}$. (Recall that s is the length variable of a path.) This speed is proportional to the tangential speed of the AGV given in equation 2.11. Notice that e is the latitude error of the AGV projected to the normal direction of the path at point $\tilde{p}$. We have the following relationship:

$$v_r \cos\theta_e = \frac{\tilde{R}_{\tilde{p}} + e}{\tilde{R}_{\tilde{p}}} \frac{ds}{d\bar{t}} \frac{d\bar{t}}{dt} \tag{2.12}$$

$$= (1 + \kappa(\bar{t})e)\frac{ds}{d\bar{t}}\dot{n} \tag{2.13}$$

Notice that the nominal time variable $\bar{t}$ can be chosen liberally, as long as it can be used to parameterize the path. One way to select $\bar{t}$ is to make it the path length variable s, that is,

$$\frac{ds}{d\bar{t}} \equiv 1 \text{ and } \bar{t} = 0 \text{ implies } s = 0$$

In this case, equation 2.13 becomes

$$\dot{s} = \frac{v_r \cos\theta_e}{1 + \kappa(s)e} \tag{2.14}$$

The latitude error itself is determined by v_r's projection onto the normal direction of $\tilde{p}$, that is,

$$\dot{e} = -v_r \sin\theta_e \tag{2.15}$$

Recall that the heading rate of the AGV is ω. And the heading rate of the virtual AGV is

$$\frac{\dot{s}}{\tilde{R}_{\tilde{p}}} = \frac{\kappa(s) v_r \cos\theta_e}{1 + \kappa(s)e} \tag{2.16}$$

Combining equation 2.14, equation 2.15 and equation 2.16, we have

$$\begin{cases} \dot{s} & = \frac{v_r \cos\theta_e}{1+\kappa(s)e} \\ \dot{e} & = -v_r \sin\theta_e \\ \dot{\theta}_e & = \omega - \frac{\kappa(s) v_r \cos\theta_e}{1+\kappa(s)e} \end{cases} \tag{2.17}$$

This is a new kinematic model with new state vector $\bar{p} = (s, e, \theta_e)$ with control input being $\bar{u} = (v_r, \omega)$. When speed is controlled by another independent controller, ω becomes the only control input.

A control law is given as below:

$$\omega = \omega_0 + \omega_1 + \omega_2 \tag{2.18}$$

where

$$\begin{cases} \omega_0 & = \frac{\kappa(s) v_r \cos\theta_e}{1+\kappa(s)e} \\ \omega_1 & = -g(e, \theta_e; t)\theta_e \\ \omega_2 & = k_e v_r \frac{\sin\theta_e}{\theta_e} e \end{cases} \tag{2.19}$$

where $g = g(e, \theta_e; t) > 0, k_e > 0$ and $v_r \neq 0$. The controller given in equation 2.18 leads to local asymptotic convergence. This can be verified by the Lyapunov function $V(e, \theta_e) = e^2 + \frac{\theta_e^2}{k_2}$. Since curvature appears in the control law, the path must be second order continuous. Because v_r can be positive or negative, this controller applies to both cases of AGV moving forwardly and moving backwardly.

If we let

$$g = k_\theta |v_r|, \ k_\theta > 0 \tag{2.20}$$

This leads to local exponential convergence with its convergence rate independent of vehicle speed. This is a property we want to have.

Recommended value for k_θ and k_e is $k_\theta = 0.75$ and $k_e = 0.25$. Notice that these values are not necessarily optimal. For details of this algorithm, see [1].

Chapter 3

Stanley Method for Cruise Control

This method is used to perform cruise control. Given a moving vehicle and a reference path, the state of its frontal wheel is denoted by

$$p_f = (x_f, y_f, \theta_f),$$

where x_f and y_f are the coordinates of frontal wheel's center point, and θ_f is its rolling direction. With a given path, there is also a unique reference path for the frontal wheel. To avoid causing confusion, we call it the reference frontal wheel path. Suppose that the nearest point on the reference frontal wheel path with respect to the frontal wheel of the AGV is $(\tilde{x}_f, \tilde{y}_f)$, and a virtual AGV is moving along the reference path with its frontal wheel being of the following state

$$\tilde{p}_f = (\tilde{x}_f, \tilde{y}_f, \tilde{\theta}_f),$$

where $\tilde{\theta}_f$ is the ideal rolling direction of the virtual AGV. We also introduce the following definition:

$$\tilde{\alpha}_f := (\cos\tilde{\theta}_f, \sin\tilde{\theta}_f) \tag{3.1}$$

$$d_f := p_f - \tilde{p}_f \tag{3.2}$$

$$e_f := d_f \times \tilde{\alpha}_f = d_{fx}\sin\tilde{\theta}_f - d_{fy}\cos\tilde{\theta}_f \tag{3.3}$$

We define the error between the rolling direction of the frontal wheel of the AGV and the rolling direction of the frontal wheel of the virtual AGV as

$$\theta_{fe} = \theta_f - \tilde{\theta}_f.$$

Like what is done in the last section, the following relation holds

$$\begin{aligned}
\dot{e}_f &= -v_f \sin(\theta_{fe}) \\
&= -v_f \sin(\theta_f - \tilde{\theta}_f) \\
&= -v_f \sin(\theta + \delta - \tilde{\theta} - \tilde{\delta}) \\
&= -v_f \sin(\delta - (\tilde{\theta} + \tilde{\delta} - \theta)) \\
&= -v_f \sin(\delta - (\tilde{\theta}_f - \theta)) \\
&= -v_f \sin(\delta - \varphi),
\end{aligned}$$

where

$$\varphi = \tilde{\theta}_f - \theta \tag{3.4}$$

If we want the latitude error of the frontal wheel diminishes with a constant rate, we need to solve the following equation with $k > 0$

$$\dot{e}_f = -v_f \sin(\delta - \varphi) = -ke_f \tag{3.5}$$

We get

$$\delta = \arcsin\left(\frac{ke_f}{v_f}\right) + \varphi$$

A drawback of this control law is that it is only legitimate when $\frac{ke_f}{v_f} < 1$. To overcome this, the control law can be relaxed to

$$\delta = \arctan\left(\frac{ke_f}{v_f}\right) + \varphi \tag{3.6}$$

This is a first order approximation of the original control law, with the cost that now it only has local exponential convergence. Because this control law does not depend on curvature of the path, it only requires that the path is first order continuous.

The convergence of the control law can be verified by substituting Equation 3.6 into Equation 3.5. By doing so, it leads to the following

$$\begin{aligned}\dot{e}_f &= -v_f \sin\arctan\frac{ke_f}{v_f} \\ &= \frac{-ke_f}{\sqrt{1 + \frac{k^2 e_f^2}{v_f^2}}}\end{aligned}$$

because of $\sin(\arctan\beta) = \frac{\beta}{1+\beta^2}$. If $\frac{ke_f}{v_f} < c > 0$, we have

$$|\dot{e}_f| > \frac{k}{\sqrt{1+c^2}}|e_f|$$

that is how fast the latitude error converges with k being positive (local exponential convergence). It is obvious that with a fixed positive k, v_f must be greater than zero.

The term φ in the control law can be interpreted as the addition of the real heading error, and a feed forward term to counterbalance curvature variance in near future. The result is that this control law is less sensitive to small control signal delay, and has less overshoot to discontinuity of curvature of the path. The cost is worse tracking error compared with the rear wheel controller in the last section.

Suggested control parameter can take the initial value of $k = 0.5$ with $v_f \leq 1\,m/s$. If quicker response is desired, the value can be increased, otherwise, the value can be decreased. The optimal value should be found by experiment for a setting-up. The details of the controller can be found in [2].

Bibliography

[1] C. Samson, "Path following and time-varying feedback stabilization of a wheeled mobile robot," *Second International Conference on Automation, Robotics and Computer Vision*, vol. 3, Jan. 1992. 5

[2] S. Thrun, M. Montemerlo, H. Dahlkamp, D. Stavens, A. Aron, J. Diebel, P. Fong, J. Gale, M. Halpenny, G. Hoffmann, K. Lau, C. Oakley, M. Palatucci, V. Pratt, P. Stang, S. Strohband, C. Dupont, L.-E. Jendrossek, C. Koelen, C. Markey, C. Rummel, J. van Niekerk, E. Jensen, P. Alessandrini, G. Bradski, B. Davies, S. Ettinger, A. Kaehler, A. Nefian, and P. Mahoney, "Stanley: The robot that won the darpa grand challenge: Research articles," *J. Robot. Syst.*, vol. 23, pp. 661–692, Sept. 2006. 7